THE PLANE PROBLEM IN THE THEORY OF NONSYMMETRICAL ELASTICITY

(PLOSKAIA ZADACHA TEORII NESIMMETRICHNOI UPRUGOSTI)

PMM Vol. 28, № 6, 1964, pp. 1117-1120

V.A.PAL'MOV
(Leningrad)

(Received July 10, 1964)

The theory of nonsymmetrical elasticity was first studied in paper [1]. A modern dirivation of the equations of the theory and an account of its basis may be found in [2 and 3].

1. To describe the state of stress in the medium, we introduce, following [1 to 3], a dyadic of couple-stresses μ in addition to the stress dyadic τ. The components of the stress dyadic represent forces acting on a unit area of the corresponding cross-section, whereas the couple-stress dyadic represents a moment acting on a unit area of the same cross-section. To describe the displacements of particles of the medium, we introduce a field of rotations Φ, which is kinematically independent of the usual displacement field $\mathbf{u}$. The stress dyadic and the couple-stress dyadic satisfy equations of equilibrium, which in the absence of body forces and moments have the form [2 and 3]

$$\nabla \cdot \boldsymbol{\tau} = 0, \qquad \nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_\times = 0 \tag{1.1}$$

Here ∇ is the differential operator of Hamilton and $\boldsymbol{\tau}_\times$ denotes the vector invariant of the stress dyadic τ [4].

We next introduce the deformation dyadics Λ and $\mathbf{M}$. For small $\mathbf{u}$ and Φ they are defined as follows [3] :

$$\Lambda = \nabla \mathbf{u} + \mathbf{I} \times \boldsymbol{\Phi}, \qquad \mathbf{M} = \nabla \boldsymbol{\Phi} \qquad (\mathbf{I} \text{ is unit dyadic}) \tag{1.2}$$

The connection between stresses and deformations for the isotropic elastic medium is given by the generalized Hooke's law

$$\boldsymbol{\tau} = \lambda \mathbf{II} \cdot\cdot \Lambda^+ + 2\mu\Lambda^+ + 2\alpha\Lambda^-, \qquad \boldsymbol{\mu} = \beta\mathbf{II} \cdot\cdot \mathbf{M}^+ + 2\gamma\mathbf{M}^+ + 2\varepsilon\mathbf{M}^- \tag{1.3}$$

which contains six elastic constants $\alpha, \beta, \gamma, \epsilon, \mu, \lambda$, whereby μ and λ are the usual Lamé constants. In the relations (1.3) a plus superscript denotes symmetric components of the dyadic and a minus superscript denotes antisymmetric components.

2. We examine an elastic body under condition of plane deformation. We set

$$\mathbf{u} = \mathbf{i}u(x, y) + \mathbf{j}v(x, y), \qquad \boldsymbol{\Phi} = \mathbf{k}\Phi(x, y) \tag{2.1}$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along the x, y, z axes of a rectangular Cartesian system of coordinates.

The nonvanishing components of the Λ and M dyadics in this case are

$$\Lambda_{xx}=\frac{\partial u}{\partial x},\qquad \Lambda_{xy}=\frac{\partial v}{\partial x}-\Phi,\qquad M_{xz}=\frac{\partial\Phi}{\partial x}$$
$$\Lambda_{yx}=\frac{\partial u}{\partial y}+\Phi,\qquad \Lambda_{yy}=\frac{\partial v}{\partial y},\qquad M_{yz}=\frac{\partial\Phi}{\partial y} \tag{2.2}$$

It is not difficult to verify that the following identities among the remaining elements of the deformation dyadics are valid (2.3)

$$\frac{\partial\Lambda_{yx}}{\partial x}-\frac{\partial\Lambda_{xx}}{\partial y}-M_{xz}=0,\qquad \frac{\partial\Lambda_{yy}}{\partial x}-\frac{\partial\Lambda_{xy}}{\partial y}-M_{yz}=0,\qquad \frac{\partial M_{yz}}{\partial x}-\frac{\partial M_{xz}}{\partial y}=0 \tag{2.4}$$

They represent the conditions of compatibility of deformations for the particular case of plane deformation. Conditions (2.3) and (2.4) coincide with the corresponding conditions of [5]. Many components of the Λ and M dyadics vanish; hence the Hooke's law relations greatly simplify

$$\tau_{xx}=\lambda(\Lambda_{xx}+\Lambda_{yy})+2\mu\Lambda_{xx},\qquad \tau_{yy}=\lambda(\Lambda_{xx}+\Lambda_{yy})+2\mu\Lambda_{yy}$$
$$\tau_{xy}=\mu(\Lambda_{xy}+\Lambda_{yx})+\alpha(\Lambda_{xy}-\Lambda_{yx}),\qquad \tau_{yx}=\mu(\Lambda_{xy}+\Lambda_{yx})-\alpha(\Lambda_{xy}-\Lambda_{yx})$$
$$\tau_{zz}=\lambda(\Lambda_{xx}+\Lambda_{yy}),\qquad \tau_{xz}=\tau_{yz}=\tau_{zx}=\tau_{zy}=0 \tag{2.5}$$
$$\mu_{xz}=(\gamma+\varepsilon)M_{xz},\quad \mu_{yz}=(\gamma+\varepsilon)M_{yz},\quad \mu_{zx}=(\gamma-\varepsilon)M_{xz},\quad \mu_{zy}=(\gamma-\varepsilon)M_{yz}$$

By virtue of (2.2) and (2.5), the components of the stress dyadic and the couple-stress dyadic do not depend on the z coordinate. Hence the equations of equilibrium reduce to the system

$$\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}=0,\quad \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}=0,\qquad \frac{\partial\mu_{xz}}{\partial x}+\frac{\partial\mu_{yz}}{\partial y}+\tau_{xy}-\tau_{yx}=0 \tag{2.6}$$

Solving the Hooke's law relationships for the components of Λ and M, we obtain

$$\Lambda_{xx}=\frac{1}{2\mu}[\tau_{xx}-\nu(\tau_{xx}+\tau_{yy})],\qquad \Lambda_{yy}=\frac{1}{2\mu}[\tau_{yy}-\nu(\tau_{xx}+\tau_{yy})] \tag{2.8}$$

$$\Lambda_{xy}=\frac{\tau_{xy}+\tau_{yx}}{4\mu}+\frac{\tau_{xy}-\tau_{yx}}{4\alpha},\qquad \Lambda_{yx}=\frac{\tau_{xy}+\tau_{yx}}{4\mu}-\frac{\tau_{xy}-\tau_{yx}}{4\alpha} \tag{2.9}$$

$$M_{xz}=\frac{1}{\gamma+\varepsilon}\mu_{xz},\qquad M_{yz}=\frac{1}{\gamma+\varepsilon}\mu_{yz},\qquad \nu=\frac{\lambda}{2(\lambda+\mu)} \tag{2.10}$$

Here ν is Poisson's ratio. The components of the stress dyadic and the couple-stress dyadic which do not enter into (2.6) to (2.10) can be expressed in terms of those components that do enter into these relations

$$\tau_{zz}=\nu(\tau_{xx}+\tau_{yy}),\quad \mu_{zx}=\frac{\gamma-\varepsilon}{\gamma+\varepsilon}\mu_{xz},\quad \mu_{zy}=\frac{\gamma-\varepsilon}{\gamma+\varepsilon}\mu_{yz} \tag{2.11}$$

Substituting Expressions (2.8), (2.9), (2.10) into (2.3) and (2.4), we obtain

$$\frac{\partial}{\partial x}\left(\frac{\tau_{xy}+\tau_{yx}}{4\mu}-\frac{\tau_{xy}-\tau_{yx}}{4\alpha}\right)-\frac{1}{2\mu}\frac{\partial}{\partial y}\left[\tau_{xx}-\nu(\tau_{xx}+\tau_{yy})\right]-\frac{\mu_{xz}}{\gamma+\varepsilon}=0 \tag{2.12}$$

$$\frac{1}{2\mu}\frac{\partial}{\partial x}[\tau_{yy}-\nu(\tau_{xx}+\tau_{yy})]-\frac{\partial}{\partial y}\left(\frac{\tau_{xy}+\tau_{yx}}{4\mu}+\frac{\tau_{xy}-\tau_{yx}}{4\alpha}\right)-\frac{\mu_{yz}}{\gamma+\varepsilon}=0 \tag{2.13}$$

$$\frac{\partial\mu_{yz}}{\partial x}-\frac{\partial\mu_{xz}}{\partial y}=0 \tag{2.14}$$

The six equations (2.6),and (2.12) to (2.14) form a complete system of equations in terms of stresses for the case of plane deformation.

Following [5 and 6], we express the stresses and the couple-stresses in terms of two stress functions

$$\tau_{xx} = \frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\psi}{\partial x\partial y}, \qquad \tau_{yy} = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\psi}{\partial x\partial y}, \mu_{xz} = \frac{\partial\psi}{\partial x} \tag{2.15}$$

$$\tau_{xy} = -\frac{\partial^2\varphi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial y^2}, \qquad \tau_{yx} = -\frac{\partial^2\varphi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial x^2}, \mu_{yz} = \frac{\partial\psi}{\partial y} \tag{2.16}$$

A straightforward calculation shows that the equations of equilibrium (2.6) and conditions (2.14) are satisfied identically.

Substitution of Expressions (2.15) and (2.16) into (2.12) and (2.13) for the determination of the stress functions φ and ψ gives

$$\frac{\partial}{\partial x}(\psi - l^2\nabla^2\psi) = -2(1-\nu)h^2\frac{\partial}{\partial y}\nabla^2\varphi \tag{2.17}$$

$$\frac{\partial}{\partial y}(\psi - l^2\nabla^2\psi) = 2(1-\nu)h^2\frac{\partial}{\partial x}\nabla^2\varphi \tag{2.18}$$

where

$$l^2 = (\gamma+\varepsilon)(^1/_4\mu + {}^1/_4\alpha), \ h^2 = (\gamma+\varepsilon)/4\mu \tag{2.19}$$

We differentiate the first equation with respect to y, the second with respect to x, and subtract one equation from the other.

We obtain a biharmonic equation for the stress function φ

$$\nabla^4\varphi = 0 \tag{2.20}$$

In a similar fashion we obtain an equation for ψ

$$\nabla^2(\psi - l^2\nabla^2\psi) = 0 \tag{2.21}$$

It should be noted that although separate equations have been obtained for φ and ψ, their solutions are nevertheless not arbitrary, but must be chosen in such a way that Equations (2.17) and (2.18) are satisfied.

We likewise note that Equations (2.17) and (2.18) coincide with those in [6] only when α is infinitely large. Equations (2.20) and (2.21) in essence coincide with the analogous equations of [5].

3. We examine the problem of the stress concentration in the neighborhood of a circular hole in a simple tensile field. Namely, we assume that the periphery of the hole $r = a$ is free of stresses and couple-stress and that at infinity we have the state of stress

$$\tau_{xx} = p, \quad \tau_{yy} = \tau_{xy} = \tau_{yx} = \mu_{xz} = \mu_{yz} = 0 \tag{3.1}$$

In the solution, we use a polar coordinate system ($x = r\cos\theta$, $y = r\sin\theta$). The stress functions φ and ψ must separately satisfy Equations

$$\nabla^4\varphi = 0, \qquad \nabla^2(\psi - l^2\nabla^2\psi) = 0 \qquad \left(\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) \tag{3.2}$$

In a polar coordinate system, Equations (2.17) and (2.18) take on the form

$$\frac{\partial}{\partial r}(\psi - l^2\nabla^2\psi) = -2(1-\nu)h^2\frac{1}{r}\frac{\partial}{\partial\theta}\nabla^2\varphi \tag{3.3}$$

$$\frac{1}{r}\frac{\partial}{\partial\theta}(\psi - l^2\nabla^2\psi) = 2(1-\nu)h^2\frac{\partial}{\partial r}\nabla^2\varphi$$

By the usual methods, we obtain the following formulas for the components of the stress dyadic in a polar coordinate system (cf. [6])

$$\tau_{rr}=\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right),\qquad \tau_{\theta\theta}=\frac{\partial^2\varphi}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right) \tag{3.4}$$

$$\tau_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right)-\frac{1}{r}\frac{\partial\psi}{\partial r}-\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2},\qquad \tau_{\theta r}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right)+\frac{\partial^2\psi}{\partial r^2}$$

$$\mu_{rz}=\frac{\partial\psi}{\partial r},\qquad \mu_{\theta z}=\frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

The boundary conditions in the problem at hand are

for $r=a$

$$\tau_{rr}=0,\qquad \tau_{r\theta}=0,\qquad \mu_{rz}=0 \tag{3.5}$$

for $r\to\infty$

$$\tau_{rr}={}^1/_2 p(1+\cos 2\theta),\quad \tau_{\theta\theta}={}^1/_2 p\,(1-\cos 2\theta),\quad \tau_{r\theta}=\tau_{\theta r}=\mu_{rz}=\mu_{\theta z}=0 \tag{3.6}$$

The stress functions satisfy conditions (3.8) and Equations (3.2) [6].

$$\varphi={}^1/_4\, pr^2\,(1-\cos 2\theta)+A\ln r+(Br^{-2}+C)\cos 2\theta \tag{3.7}$$
$$\psi=[Dr^{-2}+EK_2\,(r\,/\,l)]\sin 2\theta$$

where K_2 is the modified Bessel function of the second kind and second order.

Substitution of (3.7) into (3.3) leads to the following restriction on the coefficients in the functions φ and ψ

$$D=8\,(1-\nu)\,h^2C \tag{3.8}$$

Substituting (3.7) into (3.4) and then into the boundary conditions (3.5), we arrive, as in [6], at the following system of equations for the determination of the constants A, B, C, D, E :

$$\frac{p}{2}+\frac{A}{a^2}=0$$

$$\frac{p}{2}-\frac{4C}{a^2}-\frac{6}{a^4}(B-D)+\frac{2E}{la}\left[\frac{3l}{a}K_0\left(\frac{a}{l}\right)+\left(1+\frac{6l^2}{a^2}\right)K_1\left(\frac{a}{l}\right)\right]=0$$

$$-\frac{p}{2}-\frac{2C}{a^2}-\frac{6}{a^4}(B-D)+\frac{E}{la}\left[\frac{6l}{a}K_0\left(\frac{a}{l}\right)+\left(1+\frac{12l^2}{a^2}\right)K_1\left(\frac{a}{l}\right)\right]=0 \tag{3.9}$$

$$-\frac{2D}{a^3}-\frac{E}{l}\left[\frac{2l}{a}K_0\left(\frac{a}{l}\right)+\left(1+\frac{4l^2}{a^2}\right)K_1\left(\frac{a}{l}\right)\right]=0$$

The solution of this system, together with equation (3.8), has the form

$$A=-\frac{pa^2}{2},\quad B=\frac{pa^4\,(1-F)}{4\,(1+F)},\quad C=\frac{pa^2}{2\,(1+F)},\quad D=\frac{4\,(1-\nu)\,a^2h^2p}{1+F}$$
$$E=-\frac{palF}{(1+F)\,K_1\,(a\,/\,l)} \tag{3.10}$$

whereby

$$F=8\,(1-\nu)\,\frac{h^2}{l^2}\left[4+\frac{a^2}{l^2}+\frac{2a}{l}\,\frac{K_0\,(a\,/\,l)}{K_1\,(a\,/\,l)}\right]^{-1} \tag{3.11}$$

Using these values of the constants we find for the stress $\tau_{\theta\theta}$ on the periphery of the hole

$$\tau_{\theta\theta}=p\left(1+\frac{2\cos 2\theta}{1+F}\right)\qquad \left(\max\tau_{\theta\theta}=p\,\frac{3+F}{1+F}\ \text{ for } \theta=\pm\frac{\pi}{2}\right) \tag{3.12}$$

We introduce the stress concentration factor for the neighborhood of the hole

$$\delta = \frac{\max \tau_{\theta\theta}}{p} = \frac{3 + F}{1 + F} \tag{3.13}$$

From Formulas (3.11) and (3.13) it is seen that the stress concentration factor depends on the elastic constants of the material and the radius of the hole.

From (3.13) it is clear that the largest δ is obtained for the smallest F But from (3.11) it follows that the minimum value of F is zero and that it is attained for $l \to \infty$, i.e. for $\alpha = 0$. In this case one attains the classical value of $\delta = 3$ for the stress concentration factor.

The smallest δ is attained at the largest F. But from (3.11) it follows that the maximum F is realized for $a/l \to \infty$, $\nu = 0$, $h = l$, i.e. for $\alpha \to \infty$. For these values of the parameters we obtain $F = 2$. Thus we find

$$\delta = 5/3 \tag{3.14}$$

This is 1.8 times smaller than the classical value. For other, arbitrary possible values of the classic constants of the material, the stress concentration factor lies between the two limiting values found above.

BIBLIOGRAPHY

1. Cosserat, E. and F., Théorie des Corps Déformables, Paris, 1909.
2. Kuvshinskii, E.V. and Aero, E.L., Kontinual'naia teoriia asimmetricheskoi uprugosti. Uchet "vnutrennego" vrashcheniia (Continuum theory of axial symmetric elasticity. Effect of the "internal" rotations). Fizika Tverdogo Tela, Vol.5, № 9, 1963.
3. Pal'mov, V.A., Osnovnye uravneniia teorii nesimmetrichnoi uprugosti (Fundamental equations of the theory of nonsymmetric elasticity). *PMM* Vol.28, № 3, 1964.
4. Lagalli, M., Vektornoe ischislenie (Vector Calculus). Gostekhizdat, 1936.
5. Schaefer, H., Versuch einer Elastizitätstheorie des zweidimensionalen ebenen Cosserat-Kontinuums. Miszellanen der Angewandten Mechanik. Festschrift W.Tollimien, 1962.
6. Mindlin, R.D., Influence of Couple-stresses on Stress Concentrations. Experimental Mechanics, Vol.3, № 1, 1963.

Translated by E.E.Z.